COMPUTER EVALUATION OF NETWORK PERFORMANCE BY A TOPOLOGICAL FLOWGRAPH TECHNIQUE*

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I. ABSTRACT

Given the linear, oriented, and weighted graph of a linear active equivalent network N, a closed flowgraph can be generated. The system of linear equations represented by the flowgraph is obtained by a topological method and describes four sets of equations: the Kirchhoff voltage- and current-law equations, the voltage-current relationship equations, and the so-called control equations for the dependent generators in the network. A solution for any dependent variables in terms of the independent generator-variables can be obtained by open flowgraph techniques. For computer implementation the closed flowgraph is most suitable.

Finding network transfer functions, sensitivity and tolerance functions, consists of generation and evaluation of the flowgraph. An algorithm is presented, which generates a closed flowgraph from the linear graph of network N in the form of the flowgraph incidence matrix A_m, an array G_n which gives the gains associated with each flowgraph edge, and an array F indicating the frequency dependence of the edges in the flowgraph.

II. INTRODUCTION

From network topology it is known [1],[2],[3], that for a linear active network with dependent sources the following equation can be written.

$$\begin{bmatrix} B_f & 0 \\ 0 & Q_f \\ Y & Z \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_e \\ I_1 \\ \vdots \\ I_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -I_g(s) - E_e(s) \end{bmatrix} \qquad (1)$$

This equation expresses in matrix form Kirchhoff's voltage law (KVL), Kirchhoff's current law (KCL), the voltage-current relationship (VCR) and the control relationship (CR) equations.

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Matrix B_f is the fundamental circuit (f-circuit) matrix and Q_f is the f-cutset matrix of the linear graph corresponding to the network. Submatrices Y and Z express both the VCR and the CR equations. $\{V_1 \dots V_e I_1 \dots I_e\}$ is the variable vector for the edge voltages and currents of the linear graph (e is the total number of elements). $I_g(s)$ and $E_e(s)$ are the independent current- and voltage-generators respectively in the network. Equation (1) can be represented by a Mason flowgraph [2], [4] the properties of which are given below.

1. Properties of the Flowgraph

Each variable in the vector $\{V_1 \dots V_e I_1 \dots I_e\}$ is represented by one node. The sources defined by Mason [2],[4], are the "known variables". There are e voltage nodes (top row) and e current nodes (bottom row) in the flowgraph. The KVL matrix equation, $B_f V_e(s) = 0$, governs the relationships between the voltage nodes; and the KCL matrix equation $Q_f I_e(s) = 0$, indicates the relations between the current nodes. The VCR matrix equation gives the "vertical" node-relations in the form $V_i = Z_i I_i$ or $I_i = Y_i V_i$, where a transmittance Z_i is directed from a current node I_i to a voltage node V_i; or, similarly, Y_i is directed from V_i to I_i for the latter. The CR equation gives the generator dependence. Dependence of a dependent generator on a controlling element in the network is given by a transmittance from the controlling variable-node to the node of the controlled element. The gain of the transmittance is the "control-constant". If a variable x_j is a function of x_i: $x_j = a_{ij} x_i$, then a transmittance edge is directed from node x_i to node x_j ; the gain associated with the edge is a_{ij} . The following example will clarify the procedure for obtaining the flowgraph.

2. Example of a Flowgraph

From the equivalent network in Fig. 1, the flowgraph is to be obtained. After the linear graph has been found a tree T is chosen containing all the voltage-generators and possibly some passive elements: T = (1,4,7) where edge i is an independent voltage-generator, edges 4 and 7 are passive elements. The KVL matrix equation can now be stated:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ . \\ . \\ . \\ V_7 \end{bmatrix} = [0] \qquad (2)$$

In the flowgraph, the chord voltage in an f-circuit is expressed in terms of the branch voltages.

Similarly the KCL matrix equation, $Q_f I_e(s) = 0$, is:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_7 \end{bmatrix} = [0] \tag{3}$$

where for the flowgraph the branch current in a f-cutset is expressed in terms of the chord currents. The VCR equations are:

$$\begin{aligned} V_4 &= Z_4 I_4 & \qquad I_2 &= Y_2 V_2 \\ V_7 &= Z_7 I_7 & \qquad I_3 &= Y_3 V_3 \\ & & \qquad I_5 &= Y_5 V_5 \end{aligned} \tag{4}$$

for branches and chords respectively. The control (CR) equation yields:

$$I_6 = g_m V_4 \tag{5}$$

The flowgraph for eq. 2,3,4 and 5 is shown in Fig. 3.

The flowgraph, and the linear graph are equivalent descriptions of the same network; in the first the "dichotomous" character is explicit, because each element is represented with two variables V_i and I_i. A solution for the unknown variables in terms of the known sources is obtained by evaluating the flowgraph. Using Mason's gain formula [2],[4], the gain G can be expressed as:

$$G = x_j / x_i \tag{6}$$

where x_i is the independent source node and x_j is the variable node as shown in Fig. 4. A closed flowgraph [6]-[10] is formed, when a transmittance j is added which is directed from node x_j to node x_i (j is used as a tagging parameter [10] and

$$G = 1/j = \frac{-H(1)}{H(o)} \tag{7}$$

which is known as Shannon's gain formula [5],[9], where $H(1)$ is the summation of gains over all loopsets containing j, and $H(o)$ the summation of gains for all loopsets devoid of j. Using the closed flowgraph approach all loop sets can be found by an algorithm [14], which generates the circuits in the Mason graph and tags those circuits containing edge j. To generate the closed flowgraph from the linear graph of network N the following algorithm is developed.

III. PROCEDURE FOR OBTAINING THE CLOSED FLOWGRAPH BY DIGITAL COMPUTER

A. The steps of the procedure are

i) From a given linear active network obtain the equivalent network N .

ii) Obtain the linear, oriented, and weighted graph from N .

iii) Select all independent and dependent voltage generators as branches of tree T with possibly some passive elements to form a complete set of branches and the current generators and remaining passive elements as chords. Then number the edges according to the following rules:

a) Number all chords first in ascending order from 1 up to and including $\mu = e-v+1$;

b) Continue to number the branches from $\mu+1$ up to and including e ; and

c) Number the v nodes of the graph arbitrarily.

iv) The linear graph will be coded in matrix G, called the graph matrix, and the array A(j) (to be described in (v)), both of which are used as data input for the computer. The following is a detailed description of G:

$$G = [G_{11} \ G_{12}] \tag{8}$$

where G_{11} is of order $6 \times \mu$, and describes the properties of the chord elements, whereas G_{12} is of order $6 \times r$, $(r=v-1)$ and describes the properties of the branch elements of the graph. Entry g_{1j} of row 1 of G represents the number of the origin node of edge e_j $(j=1,2,\ldots,e)$ in the linear graph; entry g_{2j} denotes the target node of e_j (rows 1 and 2 give the orientations of the edges). Let i be the subscript number belonging to edge e_i. If entry $g_{3j} = i$ (row 3) then element e_j is a dependent generator, controlled by the current or voltage of element e_i; if $g_{3j} = j$ (the subscript number of e_j) then element e_j is a passive element; and if $g_{3j} = 0$, then e_j is an independent generator. In row 4, the numbers $1,2,\ldots,e$ are entered, referring to the edges of the linear graph. It is now shown that rows 3 and 4 together, determine what kind of element e_j is. Element e_j can be any of the following eight kinds of network elements.

Case I. If $g_{3j} = 0$ and $g_{4j} = j$, then the number j in the column corresponding to edge e_j is compared with μ where μ refers to the last column of G_{11}: if $j \leq \mu$, element e_j corresponds to a column in G_{11}. Since $g_{3j} = 0$, e_j is an independent current generator.

Case II. If $g_{3j} = 0$ and $g_{4j} = j$, but $j > \mu$, element e_j is described in G_{12} ; g_{3j} being zero, implies that e_j is an independent voltage generator.

Case III. If $g_{3j} = g_{4j} = j$ and $j \leq \mu$, e_j is in G_{11}, and is a passive element with admittance Y_j .

Case IV. If $g_{3j} = g_{4j} = j$ and $j > \mu$, e_j is described in G_{12} and is a passive element with impedance Z_j. The next four cases relate the controlled generator element e_j to the controlling generator e_i.

Case V. If $g_{3j} = i$, $g_{4j} = j$; and $i, j \leq \mu$, then e_i and e_j are described in G_{11}, and e_j is a current controlled current generator controlled by element e_i (e.g. $I_j = \beta I_i$, where both currents are related by the current gain β).

Case VI. If $g_{3j} = i$, $g_{4j} = j$, $i > \mu$, and $j \leq \mu$, then e_i is described in G_{12}, and e_j in G_{11}, implying that e_j is a voltage controlled current generator, related to e_i by a transfer admittance g_m: $I_j = g_m V_i$.

Case VII. If $g_{3j} = i$ and $g_{4j} = j$, but $i \leq \mu$ and $j > \mu$, then e_j is a current controlled voltage generator, dependent upon the current in element e_i by the transfer impedance Z_m: $V_j = Z_m I_i$.

Case VIII. If $g_{3j} = i$, $g_{4j} = j$ and $i, j > \mu$, both e_i and e_j are described in G_{12}; and e_j is a voltage controlled voltage generator dependent upon e_i by the voltage gain "a": $V_j = aV_i$. The frequency dependence of element e_j is given in row 5. Entry $g_{5j} = 1$ (-1) if the weight of edge e_j in the linear graph has the form Ks (K/s) where K is the immittance value of element e_j; and $g_{5j} = 0$ for resistive elements and generators.

Finally, row 6 is coded according to the type of network function desired as described below.

To obtain a current gain (Sec II), $1/j = I_j/I_k$, where I_j is the current variable of e_j, I_k is the independent current source and j is the edge which closes the Mason graph, set $g_{6k} = -1$ and $g_{6j} = -1$ in row 6 of G. For the transfer function $1/j = V_j/I_k$ make $g_{6k} = -1$ and $g_{6j} = 1$. (V_j is the voltage variable of e_j). For determining $Z_d = 1/j = V_k/I_k$, set $g_{6k} = -1$ and $g_{6j} = 0$. For the transfer admittance gain, $1/j = I_j/V_k$, set $g_{6k} = 1$ and $g_{6j} = -1$; for voltage gain, $1/j = V_j/V_k$, enter $g_{6k} = 1$ and $g_{6j} = 1$; and for the driving point admittance $Y_d = 1/j = I_k/V_k$ set $g_{6k} = 1$ and $g_{6j} = 0$. Note that according to the entries of row 6 of G, the algorithm closes the Mason graph with edge j.

v. The array $A(j)$ of order $1 \times e$, is the input data for the element values of the linear graph and the entries correspond to the immittances, control constants and zeros for the independent generators. The values $g_{1j}, g_{2j}, \ldots, g_{6j}$ of the graph matrix G and the j^{th} entry a_j of array $A(j)$ are punched in one card. Since one card is needed for each edge of the linear graph, e cards are punched.

vi. Finally one card is punched to enter the values of e and v (i.e. the numbers of edges and vertices) of the linear graph. The algorithm now generates the closed flow graph from the data of the e+1 cards.

B. The closed flowgraph represented by A_m, G_n and F .

a) Matrix A_m is defined as the flowgraph incidence matrix, m of order $2e \times e_f$, where e is the number of edges of the linear graph and e_f is the number of edges in the flowgraph (e current nodes and e voltage nodes sum to "2e" nodes). Entry $a_{ij} = 1(-1)$ if edge e_j is incident at node i of the flowgraph and oriented away from (toward) i; and $a_{ij} = 0$ if e_j is not incident at node i. A_m is given by

$$A_M = \left[\begin{array}{cc|c|c|c} A_{m_{11}} & 0 & & & \\ \hline 0 & A_{m_{22}} & V & C & J \end{array}\right] \qquad (9)$$

Here submatrix $A_{m_{11}}$ corresponds to the KCL equation (nodes of $A_{m_{11}}$ are the flowgraph current nodes) (see section II), $A_{m_{22}}$ to the KVL equation (nodes of $A_{m_{22}}$ correspond to the voltage nodes) and matrices V and C give the inter-nodal relationship of both the VCR and CR equations. J is the last column of A_m and represents the edge j by which the Mason graph is closed.

b) Matrix G_n as defined by

$$G_n = [G_{n_{11}} \quad G_{n_{12}} \quad G_{n_{13}} \quad 0] \qquad (10)$$

is of order $1 \times e_f$ (a row matrix). The entries of $G_{n_{11}}$ ($G_{n_{12}}$) give the gains of those edges of the flowgraph corresponding to the columns of $A_{m_{11}}$, $(A_{m_{22}})$. The entries of $G_{n_{13}}$ are the values of the immittances and control constants for the edges corresponding to the columns of submatrices V and C. The last entry of G_n is zero for edge j.

c) Matrix F as defined by

$$F = [0 \quad 0 \quad F_r \quad 0] \tag{11}$$

is of order 1 x e_f. The columns of F correspond to those of A_m. F gives the frequency dependence as obtained from row 5 of G, where the non-zero entries appear only in F_r, since edge j and the edges of $A_{m_{11}}$ and $A_{m_{22}}$ are not frequency dependent. The KCL matrix equation can be written as $Q_f I_e(s) = [Q_{f_{11}} \quad U]\, I_e(s) = 0$ [1],[2] (12)

where the columns of $Q_{f_{11}}$ correspond to the chords of the chosen tree T. The algorithm finds $Q_{f_{11}}$ from the graph matrix G. Every entry of $Q_{f_{11}}$ becomes an edge between current nodes (bottom row) of the flowgraph
Thus $A_{m_{11}}$ is obtained from $Q_{f_{11}}$ and corresponds to the KCL equation relating the current nodes of the flowgraph. The gains +1, (-1) associated with the edges corresponding to the columns of $A_{m_{11}}$ are entered into $G_{n_{11}}$, for -1, (+1) entries of $Q_{f_{11}}$. The KVL matrix equation can be written as $B_f V_e(s) = [U \; B_{f_{12}}] V_e(s) = 0$, where the column order of B_f is the same as that of Q_f so that the columns of $B_{f_{12}}$ correspond to the branches of T. Since it is well known [1],[2] that

$$Q_{f_{11}} = [-B_{f_{12}}]^T \; , \tag{13}$$

it follows that

$$A_{m_{22}} = -A_{m_{11}} \tag{14}$$

and

$$G_{n_{12}} = -G_{n_{11}} \tag{15}$$

$Q_{f_{11}}$ is generated from the first two rows of G. From rows 3 and 4 of G, the algorithm generates submatrices V and C of A_m (as outlined in III- iv). During this stage in the algorithm the gains and frequency tags of array A(j) and row 5 of G are transferred to the proper positions in $G_{n_{13}}$ and F_r respectively. Finally, from row 6 of G are determined the positions of the "+1" and "-1" of edge j in the last column of A_m. Matrices A_m, G_n and F completely describe the closed flowgraph. The following example will clarify the algorithm.

IV. EXAMPLE OF GENERATION OF A_m, G_n and F

Given the linear active network N (Fig. 5a) v_o/v_i is to be evaluated by first obtaining a closed flowgraph, by the following steps:

Step 1: Obtain the equivalent network (Fig. 5b)

Step 2: Obtain the linear graph (Fig. 5c)

Step 3: Select a tree T and number the edges of the graph following the procedure outlined previously

Step 4: Obtain the graph matrix G and array A(j) and print their values in the e input cards. Use one card to enter the number of edges e and nodes v.

$$G = \begin{bmatrix} 2 & 4 & 3 & 1 & 3 & 4 \\ 3 & 3 & 4 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A(j) = [Y_1 b \; Y_3 0 \; Z_5 \; Z_6]$$

Step 5: $Q_{f_{11}}$ is generated:

$$Q_{f_{11}} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

from which matrices $A_{m_{11}}$, $A_{m_{22}}$, $G_{n_{11}}$, and $G_{n_{12}}$ are determined.

Step 6: Submatrices V , C, $G_{n_{13}}$ and F_r are generated next.

Step 7: In the last column of A_m the incidence of edge j is defined.

Matrices A_m, G_n, and F are given as follows:

$$
A_m = \begin{array}{c|cccccc|cccccc|ccccc|c|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 1 & 1 & & & & & & & & & & & & & -1 & 1 & & \\
2 & & & 1 & & 1 & & & & & & & & & & & -1 & & \\
3 & & & & 1 & & 1 & & & & 0 & & & & & & & -1 & \\
4 & -1 & & & & & & & & & & & & & & & & & \\
5 & & -1 & -1 & -1 & & & & & & & & & 1 & & & & & \\
6 & & & & & -1 & -1 & & & & & & & & 1 & & & & \\
\hline
7 & & & & & & & -1 & -1 & & & & & & & 1 & & & \\
8 & & & & & & & & & -1 & & -1 & & & & & & & \\
9 & & & & 0 & & & & & & -1 & & -1 & & & & & 1 & \\
10 & & & & & & & 1 & & & & & & & & & & & -1 \\
11 & & & & & & & & 1 & 1 & 1 & & & -1 & & & & & \\
12 & & & & & & & & & & & 1 & 1 & & -1 & & & & 1
\end{array}
$$

$$
G_n = [\; 1 \;\; 1 \;\; 1 \;\; -1 \;\; -1 \;\; 1 \;\vdots\; -1 \;\; -1 \;\; -1 \;\; 1 \;\; 1 \;\; -1 \;\vdots\; Z_5 \;\; Z_6 \;\; Y_1 \;\; b \;\; Y_3 \;\vdots\; 0 \;]
$$

$$
F = [\; 0 \;\vdots\; 0 \;\vdots\; 0 \;\vdots\; 0 \;]
$$

and the corresponding flowgraph is shown in Fig. 6. The number of nodes of the flowgraph is equal twice e, or twice the number of edges of the linear graph. The number of edges in the flowgraph is equal to:

$e_f = 2q+d+p+1$, where
- q is the number of entries in Q_f
- d is the number of dependent generators,
- p is the number of passive elements
- 1 corresponds to j, the "unknown".

V. CONCLUSIONS

For the generation of a closed flowgraph a concise algorithm has been developed. The method presented is simpler than the method described in the literature [6]-[10], by using a simpler coded input and by making use of the duality existing between f-cutsets and f-circuit sets of the graph. The method provides all information necessary to implement the algorithm of Dunn and Chan [14], which evaluates the closed flowgraph, since this algorithm uses the non-oriented flowgraph incidence matrix A_m. Alternative evaluations of the closed flowgraph can be obtained by the procedure of Happ et al [6]-[10] or possibly by using the connection matrix C as outlined by Seshu, Hohn, and Aufenkamp [15]. A program (Fortran IV) has been written for the IBM 360 system, which generates the flowgraph as described.

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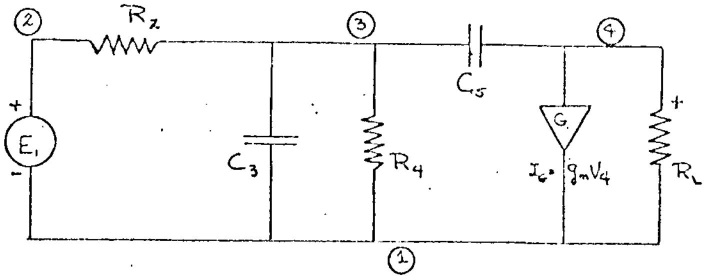
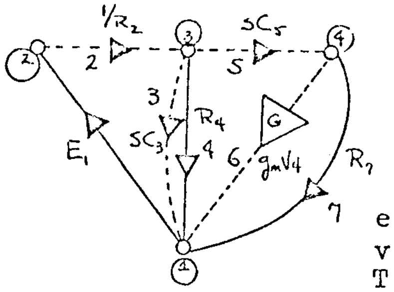


Fig. 1 Equivalent network N



e = 7
v = 4
T = (1,4,7)

Fig 2 Linear Graph

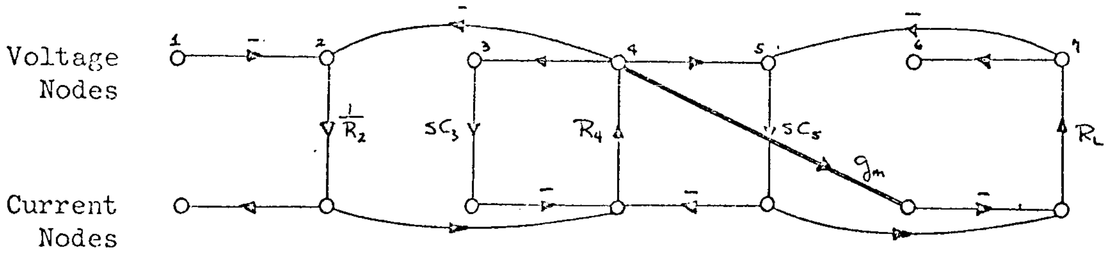


Fig 3 Flowgraph

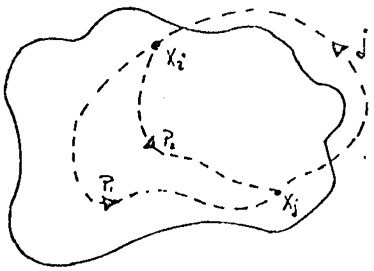
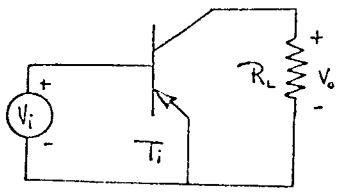


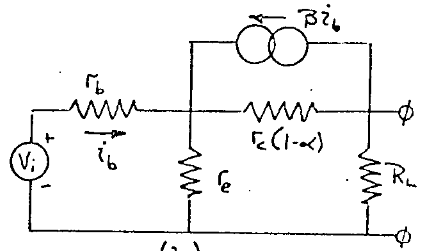
Fig 4 Flowgraph with edge j added across nodes x_i & x_j.



(a)

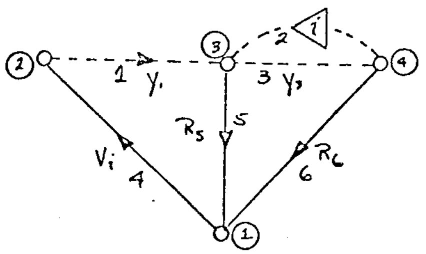
Network N

Fig 5



(b)

Equivalent Network



$I_2 = \beta I_1$

(c)

Linear graph

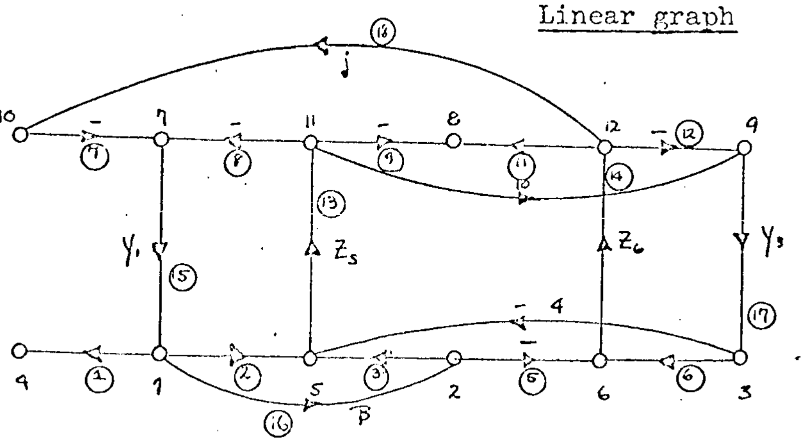


Fig 6 Flowgraph for A_m, G_n and F